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Unusual properties of superoscillating particles

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Abstract

It has been found that differentiable functions can locally oscillate on length scales that are much smaller than the smallest wavelength contained in their Fourier spectrum—a phenomenon called superoscillation. Here, we consider the case of superoscillations in quantum mechanical wavefunctions. We find that superoscillations in wavefunctions lead to unusual phenomena that are of measurement theoretic, thermodynamic and information theoretic interest. We explicitly determine the wavefunctions with the most pronounced superoscillations, together with their scaling behaviour. We also briefly address the question of how superoscillating wavefunctions might be produced experimentally.

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Introduction

Let us consider the set of differentiable functions whose Fourier transforms show only wavelengths that are larger than some finite minimum wavelength λ_{\min} . Intuitively, one may expect that none of these functions oscillates on length scales that are significantly smaller than λ_{\min} . In fact, however, it has been found that this set contains functions which locally oscillate with wavelengths that are arbitrarily smaller than λ_{\min} . The phenomenon is known as superoscillation and examples have been discussed in various contexts, from evanescent waves and seeming superluminal propagation to the trans-Planckian problem of black holes, see [1–7]. In the field of information theory, see e.g. [8, 9], the phenomenon of superoscillating signals was first observed in [10].

Our aim here is to describe and analyse phenomena that arise when quantum mechanical wavefunctions superoscillate. In particular, we will show that particles of low momentum pick up large momenta when passing through a slit, if the part of their wavefunction which passes through the slit is superoscillating. As we will explain, this phenomenon has implications

that are of measurement theoretic, thermodynamic and information theoretic interest. It will be challenging, however, to observe these effects experimentally. This is because superoscillations come at a cost: it is known that superoscillations tend to come with a relatively small amplitude.

It is, therefore, of both theoretical and practical interest to be able to calculate that wavefunction which superoscillates with the maximally possible amplitude at any pre-specified wavelength in any pre-specified interval. Here, we build on mathematical methods of [7, 10, 11] to solve this problem. In particular, we will calculate how this maximal amplitude scales when considering functions that superoscillate at higher and higher frequencies or for longer and longer intervals. We will show that the maximally possible amplitude of the most pronounced superoscillations in normalized wavefunctions decreases polynomially when increasing the frequency and decreases exponentially when increasing the total duration of the superoscillations. With the latter case we will thereby prove a conjecture of Berry [2, 3]. We will also find that this scaling behaviour possesses an information theoretic interpretation. Finally, we will speculate on a method by which, in principle, superoscillating wavefunctions might be produced experimentally.

Superoscillations

Throughout, we will consider the set of wavefunctions with momentum cutoff p_{\max} :

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-p_{\max}}^{p_{\max}} \tilde{\psi}(p) e^{\frac{ixp}{\hbar}} dp. \quad (1)$$

Note that each such ψ is differentiable. These wavefunctions are linear combinations of plane waves whose wavelengths are at least as large as $\lambda_{\min} = h/p_{\max}$. Among those wavefunctions are wavefunctions that oscillate arbitrarily quickly on arbitrarily long stretches, see [1–3]. To be precise, as was shown in [7], for any N arbitrarily chosen points $\{x_k\}_{k=1}^N$ and arbitrarily chosen amplitudes $\{a_k\}_{k=1}^N$ there exist square integrable and differentiable wavefunctions ψ which at the prescribed points take the prescribed amplitudes

$$\psi(x_k) = a_k, \quad \text{for all } k = 1, 2, \dots, N \quad (2)$$

while also obeying the momentum cutoff expressed in equation (1). In particular, we can choose the spacing of the points $\{x_k\}$ small compared to the minimum wavelength,

$$x_{k+1} - x_k \ll \lambda_{\min} \quad \text{for all } k \quad (3)$$

while choosing, for example, amplitudes a_k with alternating signs (and arbitrary moduli $|a_k|$). The wavefunctions ψ , which are guaranteed to exist by the results of [7], then contain superoscillations, in the sense that they locally oscillate faster than the shortest wavelength λ_{\min} that is contained in the wavefunction's Fourier decomposition.

Intuitively, the reason why superoscillations do not show as high frequencies in the Fourier transform is that there are subtle cancellations in the Fourier integration over all of the wavefunction, i.e. over both its superoscillating and non-superoscillating parts. In fact, it has been observed that functions that superoscillate in a given interval tend to possess some exceedingly large amplitudes immediately to the left and right of that interval, see [2, 3]. Roughly speaking, for those cancellations in the Fourier transform to take place the amplitudes in the interval with superoscillations must be small compared to amplitudes in the non-superoscillatory part of the function. Since quantum mechanical wavefunctions need to be normalized this means that if they contain any superoscillations these superoscillations can only be of small amplitude. It appears that the 'cost' for having superoscillations in a

wavefunction is the suppression of the probability for finding the particle in a region in which its wavefunction superoscillates.

The magnitude of this cost is not only of theoretical interest. It is also of significance to the question of how experiments with superoscillatory wavefunctions could be performed. In the next section, therefore, we will explicitly calculate for any given set of points $\{(x_k, a_k)\}$ that momentum-limited wavefunction ψ which passes through $\{(x_k, a_k)\}$ with the lowest cost, i.e. we calculate among all momentum-limited wavefunctions which pass through $\{(x_k, a_k)\}$ that wavefunction whose superoscillatory amplitudes after normalization are as large as possible.

Further, this ‘cost’ of superoscillations clearly also depends on the choice of points $\{(x_k, a_k)\}$. We will calculate this dependence in the later section ‘Scaling behaviour’. For example, assume we choose points $\{x_k\}$ whose spacing is small compared to λ_{\min} . This alone does not guarantee superoscillations since we may choose, e.g. all $a_k = 1$ and there would be no cost. If, however, we choose, for example, any set of amplitudes $\{a_k\}$ that are alternately positive and negative then we are considering superoscillatory wavefunctions. Their minimum cost (i.e. maximum superoscillation amplitude after normalization) depends on the moduli $|a_k|$ that one has chosen. In the section ‘Scaling behaviour’ we will calculate for each fixed set $\{x_k\}$ that choice of amplitudes $\{a_k\}$ for which this minimum cost is highest, i.e. the choice of $\{a_k\}$ which yields the superoscillations that are most pronounced, in the sense that they are the most difficult to create given the momentum limitation. Intuitively, they are those superoscillations which require the most subtle cancellations in the Fourier transform. (As we will explain below, there is no least-superoscillatory function, i.e. there is no choice of amplitudes $\{a_k\}$ for which the minimum cost would be smallest.) We will examine the scaling of the minimum cost of those most pronounced superoscillations as one either chooses the points x_k closer together or as one increases the overall number, N , of points, i.e. as one increases the frequency or the duration of the superoscillations.

Superoscillations of maximal amplitude

Throughout this section, we assume that an arbitrary set of points $\{(x_k, a_k)\}$ has been fixed. Our aim is to explicitly calculate the ‘cheapest’ wavefunction which passes through the points $\{(x_k, a_k)\}$ while being momentum-limited by equation (1). To be precise, [7] guarantees that there are wavefunctions ψ which pass through the points (x_k, a_k) while being momentum-limited by equation (1) but they are generally not normalized. After normalization, the wavelength of their superoscillations will be unchanged, of course, but the prescribed amplitudes a_k will change to the values $b_k = a_k/\|\psi\|$. Clearly, the momentum-limited wavefunction ψ which passes through the specified points (x_k, a_k) while possessing the smallest norm $\|\psi\|$ will possess the largest amplitudes $|b_k|$ after normalization, i.e. it comes with the lowest ‘cost’ in our terminology of above. Our goal is to explicitly calculate this wavefunction.

The strategy will be, first, to calculate that wavefunction ψ of the form of equation (1) which obeys the constraints of equations (2) while possessing the smallest norm. We use a variational principle for this purpose. Second, we normalize this wavefunction by dividing it by its norm. Having minimized the norm of ψ ensures that after the normalization of ψ the amplitudes $\{b_k\}$ of the normalized wavefunction’s superoscillations are maximal. (Note that this does not mean that we require the points $\{(x_k, a_k)\}$ or $\{(x_k, b_k)\}$ to be local extrema.)

Explicitly, we minimize the norm $\int_{-\infty}^{\infty} \psi^*(x)\psi(x) dx = \int_{-p_{\max}}^{p_{\max}} \tilde{\psi}^*(p)\tilde{\psi}(p) dp$ of the wavefunction ψ , subject to the constraints of equation (2), which can also be written as

$$\frac{1}{\sqrt{2\pi\hbar}} \int_{-p_{\max}}^{p_{\max}} \tilde{\psi}(p) e^{\frac{ix_k p}{\hbar}} dp = a_k. \quad (4)$$

Thus, with Lagrange multipliers μ_r , the Lagrangian of the variational problem reads

$$L = \int_{-p_{\max}}^{p_{\max}} \tilde{\psi}^*(p) \tilde{\psi}(p) dp + \sum_{r=1}^N \frac{\mu_r}{\sqrt{2\pi\hbar}} \int_{-p_{\max}}^{p_{\max}} \tilde{\psi}(p) e^{\frac{ix_r p}{\hbar}} dp + \text{c.c.},$$

from which we obtain

$$\tilde{\psi}(p) + \sum_{r=1}^N \frac{\mu_r^*}{\sqrt{2\pi\hbar}} e^{-\frac{ix_r p}{\hbar}} = 0. \quad (5)$$

Using equation (4), we have, therefore:

$$\frac{-1}{2\pi\hbar} \sum_{r=1}^N \mu_r^* \int_{-p_{\max}}^{p_{\max}} e^{\frac{ip}{\hbar}(x_k - x_r)} dp = a_k. \quad (6)$$

We define the symmetric matrix $\{S_{k,r}\}_{k,r=1}^N$ through

$$\begin{aligned} S_{k,r} &= \frac{1}{2\pi\hbar} \int_{-p_{\max}}^{p_{\max}} e^{\frac{ip}{\hbar}(x_k - x_r)} dp \\ &= \frac{\sin((x_k - x_r)p_{\max}/\hbar)}{\pi(x_k - x_r)}. \end{aligned} \quad (7)$$

The matrix S is in fact invertible. To see this, note that it is positive definite:

$$\sum_{r,k=1}^N v_r^* S_{r,k} v_k = \frac{1}{2\pi\hbar} \int_{-p_{\max}}^{p_{\max}} \left| \sum_{l=1}^N v_l e^{ipx_l} \right|^2 dp. \quad (8)$$

Equation (6) now takes the form

$$-\sum_{r=1}^N S_{k,r} \mu_r^* = a_k, \quad (9)$$

and we can solve for the coefficients μ_r^* :

$$\mu_r^* = -\sum_{m=1}^N S_{r,m}^{-1} a_m. \quad (10)$$

Using equation (5), we finally obtain the desired wavefunction in the momentum space representation:

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \sum_{r=1}^N \sum_{m=1}^N S_{r,m}^{-1} a_m e^{-\frac{ix_r p}{\hbar}}. \quad (11)$$

In the position representation, the superoscillating wavefunction reads

$$\begin{aligned} \psi(x) &= \frac{1}{2\pi\hbar} \int_{-p_{\max}}^{p_{\max}} \sum_{r=1}^N \sum_{m=1}^N S_{r,m}^{-1} a_m e^{\frac{i(x-x_r)p}{\hbar}} dp \\ &= \sum_{r,m=1}^N S_{r,m}^{-1} a_m \frac{\sin((x-x_r)p_{\max}/\hbar)}{\pi(x-x_r)}. \end{aligned} \quad (12)$$

We see that ψ is a delicate linear combination of shifted copies of the function $\sin(2\pi x/\lambda_{\min})/x$ each of which is of only slow variation. Among all wavefunctions which obey equation (1) and which take the prescribed amplitudes a_k at the N prescribed positions x_k , the wavefunction

given in equation (12) possesses the smallest norm. Thus, the normalized wavefunction $\psi^{(n)} = \frac{1}{\|\psi\|} \psi$ superoscillates with the largest achievable amplitudes:

$$\psi^{(n)}(x_k) = \frac{a_k}{\|\psi\|}, \quad \text{for all } k = 1, \dots, N. \quad (13)$$

In other words, among all normalized wavefunctions obeying the momentum cutoff of equation (1) and passing through the points (x_k, ca_k) for some positive c , the wavefunction $\psi^{(n)}$ possesses the largest possible c .

Acceleration through a single slit

Consider a low-momentum particle which passes through a slit in a screen. The particle's position in the direction parallel to the screen thereby becomes determined to within the width L of the slit. By the uncertainty principle, this implies that the particle's momentum Δp parallel to the screen becomes uncertain, such as to obey $\Delta p \geq \frac{h}{L}$. Thus, as is well known, a particle may acquire momentum when passing through a narrow slit. We can now see, however, that the uncertainty principle is not the only reason why particles can acquire momentum when passing through a slit.

Namely, consider an incident low-momentum particle whose wavefunction in the direction parallel to the screen possesses spatial superoscillations just where the wavefunction meets the slit in the screen. In this case, the wavefunction which emerges from the slit is spatially oscillating with the very short wavelength of the superoscillations where the slit is, and is zero elsewhere. The emerging wavefunction's rapid oscillations now do show as high frequencies in its Fourier transform. This is because the emerging wave no longer possesses the high-amplitude non-superoscillating parts which cancelled the high frequencies' occurrence in the incident wave's Fourier transform. Thus, the emerging quantum particle acquires a correspondingly high momentum expectation value.

Note that since the superoscillating stretch of the particle's wavefunction can be made arbitrarily wide the slit need not be narrow. In addition, we know that the wavelength of the superoscillations in the slit interval can be made arbitrarily short. It is possible, therefore, to arrange the momentum uncertainty of the emerging particles to be small, i.e. close to the lower bound h/L , while their acquired momentum expectation value can be chosen arbitrarily larger than h/L . Explicit examples of this are calculated in a follow-up paper [12].

Scaling behaviour

In the previous section, we referred to the fact that one can always find momentum-limited wavefunctions which superoscillate on an arbitrarily large interval with arbitrarily short wavelength. Let us now consider the associated cost. The cost is, of course, that even the superoscillations of maximal amplitude that we calculated above will be of smaller and smaller amplitude when we increase the frequency and/or the number of required superoscillations. In order to find this scaling behaviour we need to calculate the maximum amplitude of superoscillations in normalized wavefunctions as a function of the choice of points $\{(x_k, a_k)\}$.

We already mentioned that it had been observed that the more superoscillations one requires, the larger the amplitudes immediately to the left and right of the interval with superoscillations tend to be. Berry, see [2], conjectured that this 'cost' generally scales exponentially with the length of the interval in which a function is superoscillating.

Quantum mechanical wavefunctions, in particular, need to be normalized, i.e. if the wavefunction ψ passes through prescribed points $\{(x_k, a_k)\}$ then the normalized wavefunction

$\psi^{(n)}$ passes through the points $\{(x_k, a_k/\|\psi\|)\}$. Berry's conjecture thus translates into the statement that the maximum amplitudes b_k of superoscillations which occur in normalized wavefunctions should decrease exponentially when requiring a larger and larger number of superoscillations.

We will determine the exact scaling behaviour of the norm $\|\psi\|$ with respect to not only the duration, but also with respect to the wavelength of the prescribed superoscillations, i.e. with respect to the choice of points $\{(x_k, a_k)\}$. Recall that ψ is here that wavefunction which passes through a given set $\{(x_k, a_k)\}$ while possessing the minimum possible norm. It is of course given by equation (12) and it is the wavefunction of the maximum possible superoscillatory amplitudes b_k after normalization. Now in order to determine the scaling behaviour of the norm $\|\psi\|$ note that, from equation (5), the norm obeys

$$\begin{aligned}\|\psi\|^2 &= \frac{1}{2\pi\hbar} \int_{-p_{\max}}^{p_{\max}} \left| \sum_{r=1}^N \mu_r^* e^{\frac{-ix_r p}{\hbar}} \right|^2 dp \\ &= \sum_{k,r=1}^N \frac{\mu_r^* \mu_k}{2\pi\hbar} \int_{-p_{\max}}^{p_{\max}} e^{\frac{i(x_k - x_r)p}{\hbar}} dp \\ &= \vec{\mu}^\dagger S \vec{\mu}.\end{aligned}$$

We used vector notation: $\vec{\mu} = \{\mu_r\}_{r=1}^N$. Using equation (10), we obtain

$$\|\psi\|^2 = \vec{a}^\dagger S^{-1} \vec{a}. \quad (14)$$

Equation (14) expresses the smallest norm of any momentum-limited wavefunction ψ which passes through the prescribed points $\{(x_k, a_k)\}$ for $k = 1, 2, \dots, N$. We will be interested in how that norm increases when increasing the superoscillations' frequency or duration, i.e. when choosing the x_k closer to another or when increasing the number N of prescribed points.

First, however, we need to choose the values $\{a_k\}$. We can choose them to be oscillating in various ways, or even not oscillating at all—with the associated cost, i.e. the minimum norm $\|\psi\|$, being correspondingly larger for some choices and smaller for other choices of the $\{a_k\}$. For example, if the distance between successive x_k is smaller than h/p_{\max} then one choice which is sure to give us superoscillations is to choose alternating amplitudes $a_k = (-1)^k$. We expect this choice to come with some considerable cost. Interestingly, this is in general not the most norm-expensive choice, i.e. in this sense it is generally not the most pronounced superoscillatory choice. For any given set $\{x_k\}$ there are generally other choices of amplitudes $\{a_k\}$ for which more fine-tuning is required to pass a momentum-limited wavefunction through the points $\{(x_k, a_k)\}$.

To see this note that, from equation (14), it is most norm expensive to choose the set $\{a_k\}$ such that when read as a vector, \vec{a} , it is an eigenvector of the symmetric matrix S^{-1} with the largest eigenvalue. Numerically, we find that the coefficients a_k of those eigenvectors tend to be of alternating sign with moduli $|a_k|$ that are small for k close to 1 and N and large for intermediate values of k , much like a wave packet. For example, let us consider the case where the minimum wavelength is ten units long, $\lambda_{\min} = 10$, while we specify, say, $N = 6$ amplitudes at unit spacing: $x_k = k$ for $k = 1, 2, \dots, 6$. The smallest eigenvalue of the matrix S is then $s_{\min} = 3.83 \times 10^{-8}$ and the corresponding eigenvector is $\vec{a} = (0.07, -0.33, 0.62, -0.62, 0.33, -0.07)$, here given to three significant digits of precision. Intuitively, this means that while it always requires fine-tuning to have a momentum-limited wavefunction oscillate faster than the minimum wavelength h/p_{\max} in some interval, it requires the most fine-tuning to have a momentum-limited wavefunction which does this while possessing also the envelope of a wave packet in that interval.

We will call those wavefunctions the wavefunctions with the most pronounced superoscillations. Our aim is to calculate the scaling behaviour of their cost, i.e. of their norm. Recall that this means that we aim to calculate how fast the maximum amplitude of superoscillations in normalized wavefunctions decreases as we increase N or place the x_k closer together—while in each case choosing the amplitudes $\{a_k\}$ so as to obtain the most pronounced (i.e. most difficult to achieve) superoscillations. Of course, wavefunctions with less pronounced superoscillations or no superoscillations come cheaper. Note, however, that there is no cheapest superoscillatory wavefunction that we could consider, i.e. there is no choice of amplitudes $\{a_k\}$ for which the minimum cost of superoscillations is the smallest. This is because one can always lower the minimum cost by choosing sets of amplitudes $\{a_k\}$ that are less and less superoscillatory—until one obtains normal-oscillatory behaviour at no cost.

Our aim now is to calculate the scaling behaviour of those in this sense most pronounced superoscillating wavefunctions and prove Berry's conjecture in this case. We know already that these wavefunctions are obtained when \vec{a} is chosen to be an eigenvector of the symmetric matrix S with the smallest eigenvalue, s_{\min} . That wavefunction ψ which possesses the maximal amplitudes after normalization then superoscillates, as $\psi^{(n)}$, with the amplitudes:

$$\psi^{(n)}(x_k) = s_{\min}^{1/2} \frac{a_k}{\|\vec{a}\|}. \quad (15)$$

For simplicity, let us prescribe the superoscillating amplitudes at equidistant points x_k with spacing Δx , namely $x_k = k\Delta x$ for $k = 0, \dots, (N-1)$. In this way, we obtain a matrix S which is of the form of a so-called prolate matrix. We can benefit from the fact that the eigenvalues of prolate matrices were studied in [11]: the definition of the prolate matrix $\rho(N, W)$ in equation (21) of [11] matches ours through $S_{r,k} = \rho_{r,k}(N, W(\Delta x))/\Delta x$ with $W(\Delta x) = \Delta x p_{\max}/h$.

We can now use equation (64) of [11] to obtain the scaling of s_{\min} , and thus of the $b_k = \psi^{(n)}(x_k)$, for fixed N and decreasing spacing Δx :

$$\psi^{(n)}(x_k) \propto s_{\min}^{1/2} \propto (\Delta x)^{N-1}. \quad (16)$$

Thus, if a fixed number $N/2$ of equidistant superoscillations is compressed into a smaller region, i.e. if the superoscillations' wavelength $2\Delta x$ is reduced, then the amplitude of the most pronounced superoscillations decreases polynomially with the wavelength of the superoscillations.

In the case where the superoscillation wavelength, i.e. the spacing Δx , is held fixed and the number N of superoscillations is increased, we can use equations (13), (58) from [11] to readily find the scaling behaviour of the smallest eigenvalue s_{\min} of S for large N , to obtain

$$\psi^{(n)}(x_k) \propto s_{\min}^{1/2} \propto N^{1/4} e^{-\gamma N/2}. \quad (17)$$

Here, γ is positive and depends on Δx but not on N . Thus, we proved that the amplitude of the most pronounced superoscillations indeed decreases exponentially with the number of superoscillations, i.e. with the length of the superoscillating stretch.

Note that while these scaling results imply exceedingly small superoscillation amplitudes b_k for the incident wavefunction it is clear that whenever the particle with the superoscillating stretch of its wavefunction does pass through the slit, then the emerging superoscillations' amplitudes are boosted up to order one through the renormalization of the collapsed wavefunction.

Open questions

The phenomenon of superoscillations raises a number of basic questions, which we here only begin to address.

(1) Our finding that the superoscillations' amplitude decreases exponentially with the length of the superoscillating stretch possesses an information theoretic interpretation after translation into the language of communication theory. Namely, instead of wavefunctions with a finite momentum cutoff, let us consider the mathematically equivalent case of signals with a finite bandwidth. We see that, by using superoscillating signals, it is in principle possible to encode arbitrary amounts of information into any arbitrarily short interval of a low-bandwidth signal. This signal is then able to pass through any channel of correspondingly low bandwidth. Every channel has some level of noise, however, and this is where the cost arises: as Shannon showed in [8], signals which pass through a channel of bandwidth ω_{\max} can deliver information at most at a rate $\omega_{\max} \log_2(1 + S/N)$, where S/N is the signal-to-noise ratio. Thus, superoscillatory information compression in signals is possible to an arbitrary extent—but the associated cost is that the required signal power grows exponentially with the length of the part of the message that is superoscillatory. For wavefunctions, this neatly corresponds to our finding that the norm of maximally superoscillatory wavefunctions grows exponentially with the number of prescribed superoscillations. It would be interesting to analyse, analogously, the scaling behaviour and information theoretic interpretation of classes of less than maximally superoscillatory wavefunctions but we will not pursue this here. It should also be most interesting to generalize this result on the scaling of superoscillations to the case of fields with finite information density in curved space, see [13].

(2) Normally, if the outcome of a conditional measurement can be predicted it is thought to be consistent to assume that the particle possessed the property in question already before the conditional measurement, at least with some probability. For example, let us consider the case of a conditional momentum measurement: first we measure if the particle is in a specified region of size L and if yes then we perform a momentum measurement. Assume the particle is initially in a state so that it can be predicted that this conditional momentum measurement will yield with high probability a momentum close to some specific large value p_{pred} . Given this information one might be tempted to assume that already in its initial state the particle must have possessed such a large momentum p_{pred} at least with some finite probability.

We have now seen, however, that this assumption is not justified: we can arrange a realization of the above setup through a particle whose initial wavefunction is momentum-limited by a value p_{\max} obeying $p_{\max} < p_{\text{pred}}$ and is superoscillating in the specified region at the rate corresponding to p_{pred} . In this case, we can predict that if the particle is found in the specified region, and if then its momentum is measured, it will be found to possess a momentum close to p_{pred} . But we know also that before the position and momentum measurements the particle had zero probability for the momentum p_{pred} .

Of course, the phenomenon is entirely due to the momentum-changing effect of the position measurement that determines whether or not the particle is in the specified region. It is noteworthy, however, that the minimum amount of momentum uncertainty $\Delta p = \hbar/L$ introduced by the position measurement can be made arbitrarily small. This is because the setup of above can be realized with the specified region being of arbitrarily large finite size L since there is no limit to how large the superoscillating stretch of a wavefunction can be made. This issue is further investigated in the follow-up paper [12] where concrete examples of such initial wavefunctions are calculated.

(3) In the single slit setup of above, those superoscillating particles which do pass through the slit will gain, with little uncertainty, arbitrarily predetermined amounts of momentum and

energy. The energy picked up by those particles is available to do work because the particle's motion after the screen is largely in a predetermined direction. Due to momentum and energy conservation, this gain must be balanced by a corresponding loss in the momentum and energy of the screen. Thus, those particles are accelerated by strong interactions with the slit walls, thereby in effect cooling the screen. It is not obvious and should be very interesting to explore if, in an ensemble of experimental runs, this cooling of the screen is offset by the heating of the screen due to the impact of those particles which miss the slit. Otherwise, consistency with the second law of thermodynamics would appear to imply that it is necessarily entropy-expensive to produce superoscillating wavefunctions.

(4) Further, the rate at which incident particles pass through the slit and thereby take energy from the screen does not depend on the screen's temperature. How, therefore is it ensured that the screen always has sufficient energy and momentum available for those particles which happen to pass through the slit? Although we do not prove this here, the answer should be provided by the uncertainty relation applied to the position and momenta of the slit walls themselves—analogue to Feynman's explanation of the momentum balance in the double slit experiment in [14]. Concretely, if the position of a cold screen is known sufficiently accurately to ensure that it is the superoscillating part of incident wavefunctions which passes through the slit, then by the uncertainty relation the screen possess sufficient momentum uncertainty to be able to provide the required momentum to the superoscillating particles which pass through the slit. Conversely, if the screen is known to possess very little momentum, then by the uncertainty relation the position of the screen and its slit cannot be known with sufficient precision to ensure for the emerging particles that it was the superoscillating part of their wavefunction which passed through the slit.

(5) The question arises of how superoscillating wavefunctions might be produced experimentally. For example, can we design a potential $V(x)$ whose ground state has a superoscillating stretch? One possibility might be to start with a harmonic oscillator potential centred about $x = 0$. Its ground state's momentum range is effectively cut off at high momenta since it behaves as $\propto e^{-kp^2}$. Let us add to the potential several relatively sharp spikes in close proximity, close to $x = 0$. The ground state's wavefunction should then become quickly varying in the region where the spikes are, while also suppressed in amplitude. Whether this would effectively constitute a superoscillatory ground state remains to be studied.

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